

On the consistent solution of the gap-equation for spontaneously broken $\lambda\Phi^4$ -theory

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Abstract

We present a self-consistent solution of the finite temperature gap-equation for $\lambda\Phi^4$ theory beyond the Hartree-Fock approximation using a composite operator effective action. We find that in a spontaneously broken theory not only the so-called daisy and superdaisy graphs contribute to the resummed mass, but also resummed non-local diagrams are of the same order, thus altering the effective mass for small values of the latter.

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INTRODUCTION

The study of Φ^4 – theory at finite temperature is of great interest for a wide field of applications. A self-interacting scalar field serving as a simple model for the Higgs particle in the standard model of electroweak interactions may allow the study of symmetry changing phase transitions. In fact, the order of the electroweak phase transition plays a crucial rôle in the framework of cosmological scenarios as well as for the badly understood process of baryogenesis [1]. Despite the simplicity of the model, one may at least hope to gain some insight in the mechanism of the phase transition. Moreover, the theory is a suitable test ground for analytic non-perturbative methods, e.g. variational methods [2] as well as for lattice simulations [3].

High temperature symmetry restoration in a spontaneously broken theory was already noted by Kirzhnits and Linde [4] and worked out quantitatively subsequently [5]. The convenient tool to study the behavior of the theory turns out to be the effective potential. Whereas the critical temperature is already determined by the one loop potential and thus relatively simple to find, the order of the phase transition depends on the detailed shape of the potential which requires also the analysis of higher loop contributions, even for small coupling constant.

In particular, one finds, quantizing the theory around the classical non-trivial minimum that the one loop self-energy at high temperature behaves like $(m_T)^2 \sim (\sqrt{\lambda}T)^2$, independent of external momenta and the value of the classical minimum. Thus, for large T the thermal mass dominates over the tree-level mass and the minimum of the effective action becomes the trivial one. The theory exhibits two important features: non-temperature stable vacuum and effective temperature dependent mass. It was consequently proposed [2] to move the tree level mass into the interaction part and to start perturbation theory with the free Lagrangian $\frac{1}{2}\Phi(\square + \Omega^2)\Phi$ including a yet undetermined mass parameter Ω . Then one calculates the effective potential and fixes the parameter by the 'principle of minimal sensitivity' $\partial V(\Phi_{min})/\partial\Omega = 0$.

A more systematic approach of a self-consistent loop expansion was developed already some time ago by Cornwall, Jackiw, and Tomboulis (CJT) in their effective action formalism for composite operators [6]. The basic idea is to introduce a bilocal mass operator in the generating functional instead of the local mass Ω . Then one defines the generalized effective action Γ as the double Legendre transform of the generating functional which is now not only a functional of the expectation value of the field but also depends on the expectation value of the time-ordered product $T\Phi(x)\Phi(y)$. The principle of minimal sensitivity gets replaced by the so-called gap-equation $\delta\Gamma(\Phi, G)/\delta G(x, y) = 0$ which is employed to eliminate the exact two-point Green function $G(x, y)$.

This paper is dedicated to investigate on the solution of the gap-equation beyond the Hartree-Fock approximation. Whereas the latter amounts to simply include only a local mass term in the ansatz for the Green function [7,8], thus being a slightly more elegant reformulation of the variational approaches mentioned above, we will consistently [9] include also the non-local contribution appearing in a spontaneously broken theory.

Our approach is as follows. We split the inverse Green function into a local mass term absorbing the usually resummed graphs, and a non-local self-energy part. By this ansatz, the gap-equation is recast into a non-linear integral equation for the self-energy, which we solve approximately by a suitably chosen ansatz. Self consistency requires the unknown mass parameter to satisfy an 'effective-mass' equation, the latter differing from the one found for the effective mass in the superdaisy resummation by an additional term which dominates in the limit of vanishing mass parameter. We discuss the behavior of the mass parameter near the critical temperature. We lay great emphasis on dealing with the divergencies correctly.

CJT COMPOSITE OPERATOR FORMALISM

It is useful to briefly review the basic steps in the construction of the generalized effective action for a classical action $I(\Phi)$ [6]. One starts with the generating functional in Euclidean

space-time

$$Z(J, K) = \int \mathcal{D}\oplus \exp - \left[\mathcal{I}(\oplus) + \int_{\S} \mathcal{J}(\S)\oplus(\S) + \frac{\infty}{\epsilon} \int_{\S} \int_{\dagger} \oplus(\S)\mathcal{K}(\S, \dagger)\oplus(\dagger) \right]$$

and $W(J, K) = \log Z(J, K)$. After defining the classical field $\phi(x)$ and the Green function $G(x, y)$ by

$$- \frac{\delta W(J, K)}{\delta J(x)} = \phi(x) = \langle \Phi(x) \rangle_{J, K}, \quad - \frac{\delta W(J, K)}{\delta K(x, y)} = \frac{1}{2}(G(x, y) + \phi(x)\phi(y)) \quad (1)$$

one performs a Legendre transformation to find the generalized effective action

$$\Gamma(\phi, G) = W(J, K) + \int_x \phi(x)J(x) + \frac{1}{2} \int_x \int_y (\phi(x)K(x, y)\phi(y) + G(x, y)K(x, y))$$

where J, K at the r.h.s. have to be taken at the solution of Eq. (1). From the above definition, one immediately deduces the inverse relations

$$\frac{\delta \Gamma(\phi, G)}{\delta \phi(x)} = J(x) + \int_y K(x, y)\phi(y), \quad \frac{\delta \Gamma(\phi, G)}{\delta G(x, y)} = \frac{1}{2}K(x, y).$$

The last equation tells us that if one wants back the standard effective action defined by $K(x, y) = 0$ one has to impose the so-called 'gap-equation'

$$\frac{\delta \Gamma(\phi, G)}{\delta G(x, y)} = 0.$$

In order to give a series expansion of $\Gamma(\phi, G)$ we define the functional operator

$$D^{-1}(\phi; x, y) = \frac{\delta^2 I(\phi)}{\delta \phi(x) \delta \phi(y)}$$

and one finds

$$\Gamma(\phi, G) = I(\phi) + \frac{1}{2} \text{Tr} \ln(D_0 G^{-1}) + \frac{1}{2} \text{Tr}(D^{-1} G - 1) + \Gamma^{(2)}(\phi, G),$$

$D_0(x, y)$ being the free propagator derived from the part of the action which is quadratic in the fields. The quantity $\Gamma^{(2)}(\phi, G)$ contains all two-loop contributions and higher and has to be calculated as follows: Shift the field Φ in the classical action by ϕ . Then $I(\Phi + \phi)$ contains terms cubic and higher in Φ which define the vertices. $\Gamma^{(2)}(\phi, G)$ is given by all

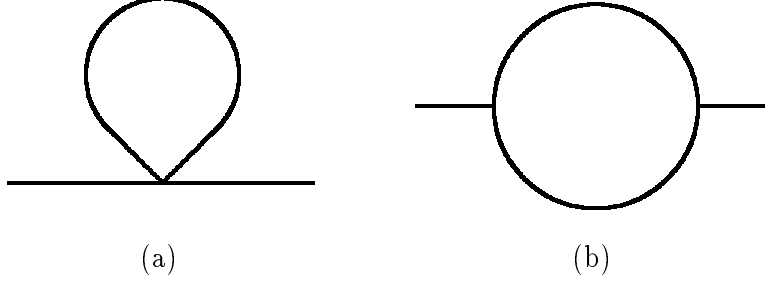


FIG. 1. Feynman graphs contributing to the lowest order self-energy. Lines denote the full Green function $G(x, y)$, the tree-vertex is given by $\lambda\phi/3!$ and the four-vertex by $\lambda/4!$.

two-particle-irreducible vacuum graphs with the given set of vertices and the propagators replaced by $G(x, y)$. The gap-equation reads

$$G^{-1}(x, y) = D^{-1}(x, y) + 2 \frac{\delta \Gamma^{(2)}(\phi, G)}{\delta G(x, y)}.$$

$\lambda\Phi^4$ -THEORY

The self-energy graphs we are going to resum are shown in Fig. 1. Although the non-local contribution Fig. 1b is formally of order λ^2 it contributes to order λ in the false vacuum since already the tree-level value of the expectation value of the field is $\phi \sim m/\sqrt{\lambda}$. Consequently, the three-vertex $\lambda\phi/3!$ is of order $\sqrt{\lambda}$ which means that in the case of a spontaneously broken theory the non-local contribution (b) is as important as the local one (a). We will show that this is also true at finite temperature. Thus, the Hartree-Fock approximation involving only the local graph which corresponds to the superdaisy resummation at finite temperature is incomplete. Including the non-local graph Fig. 1b in the self-energy, the gap-equation in Fourier-space reads

$$G^{-1}(\vec{k}, \omega) = \vec{k}^2 + \omega^2 + \overbrace{(-m^2 + \frac{\lambda}{2}\phi^2) + \frac{\lambda}{2}T \sum_{\omega'} \int \frac{d^3p}{(2\pi)^3} G(\omega', \vec{p})}^{M^2} + \frac{1}{2}(\lambda\phi)^2 T \sum_{\omega'} \int \frac{d^3p}{(2\pi)^3} G(\omega, \vec{p}) G(\omega - \omega', \vec{p} - \vec{k}) \quad (2)$$

where we use imaginary time formalism with $\omega = 2\pi nT$, $n \in \mathbf{Z}$. The first sum/integral corresponding to graph (a) does not depend on external momenta and energies and thus can

be treated as mass term. We employ the ansatz

$$G^{-1}(\omega, \vec{k}) = \vec{k}^2 + \omega^2 + \mu^2 + \Pi(\omega, \vec{k}) \quad (3)$$

and subsequently absorb all local contributions in the mass parameter μ which has to be determined self-consistently afterwards. Plugging (3) into (2), the gap-equation transforms into an equation for $\Pi(\omega, \vec{k})$,

$$\begin{aligned} \Pi(\omega, \vec{k}) = M^2 - \mu^2 + \frac{1}{2}(\lambda\phi)^2 T \sum_{\omega'} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\vec{p}^2 + \omega'^2 + \mu^2 + \Pi(\omega', \vec{p})} \times \\ \times \frac{1}{(\vec{k} - \vec{p})^2 + \mu^2 + (\omega - \omega')^2 + \Pi(\omega - \omega', \vec{k} - \vec{p})}. \end{aligned} \quad (4)$$

Simple power counting now reveals that the expression involving the integration is logarithmic divergent for large momenta \vec{p} . To isolate these divergencies, we subtract and add a term of the form

$$T \sum_{\omega'} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\vec{p}^2 + \omega'^2} \frac{1}{\vec{p}^2 + \omega'^2 + \mu^2} \quad (5)$$

which in the high temperature ($T \gg \mu$) limit evaluates to

$$\frac{T}{4\pi\mu} - \mathcal{D}, \quad \mathcal{D} = \frac{1}{8\pi^2}(1 - \gamma + \log 2\pi + \log \frac{T}{\Lambda}), \quad (6)$$

where we introduced a UV-cutoff Λ . The difference between the last summand in Eq. (4) and the subtraction (5) is a UV-finite quantity. Consequently, in the high temperature limit, we are allowed to pick only the zero-mode in the thermal sum in the difference (4) - (5). The $\omega' = 0$ contribution of the subtraction exactly cancels the first term in (6), leaving us with $\frac{1}{2}\lambda^2\phi^2\mathcal{D}$ as divergent part of the r.h.s. of the gap-equation (4). It is now possible to absorb the logarithmic divergence in the mass renormalization and we choose the parameter μ such that

$$M^2 - \mu^2 - \frac{\lambda^2\phi^2}{2}\mathcal{D} = 0 \quad (7)$$

which renders the self-energy $\Pi(\omega, \vec{k})$ in equation (4) a finite quantity. Furthermore, one may simplify equation (4) by considering the $\omega = 0$ mode of the self-energy only since the non-static modes are at least suppressed by a factor $1/T^2$. Thus putting $\Pi(\omega = 0, |\vec{k}|) = \Pi(k)$ we end with the integral equation

$$\hat{\Pi}(\hat{k}) = \kappa \int_0^\infty d\hat{p} \hat{p}^2 \int_{-1}^1 \frac{dz}{(\hat{p}^2 + 1 + \hat{\Pi}(\hat{p}))(\hat{p}^2 + \hat{k}^2 + 2\hat{p}\hat{k}z + 1 + \hat{\Pi}(\sqrt{\hat{p}^2 + \hat{k}^2 + 2\hat{p}\hat{k}z}))} \quad (8)$$

where we introduced the dimensionless quantities $p = \mu\hat{p}$, $\Pi(p) = \mu^2\hat{\Pi}(\hat{p})$ and the parameter $\kappa = T(\lambda\phi)^2/(8\pi^2\mu^3)$.

SOLUTION OF THE GAP-EQUATION

Without going in the details of solving (8) one can already make some useful statements about the behavior of $\hat{\Pi}(\hat{k})$. Firstly, the denominator of (8) is a positive definite quantity, thus $\hat{\Pi}(\hat{k}) \geq 0$. Furthermore, assuming that the solution is bound, i.e. $\hat{\Pi}(\hat{k}) \leq \hat{\Pi}_{max}$, the denominator may be estimated from below (above) by $(\hat{p}^2 + 1)(\hat{p}^2 + \hat{k}^2 + 2\hat{p}\hat{k}z + 1)$ and $(\hat{p}^2 + 1 + \hat{\Pi}_{max})(\hat{p}^2 + \hat{k}^2 + 2\hat{p}\hat{k}z + 1 + \hat{\Pi}_{max})$ respectively. Carrying out the integration on the r.h.s. in (8), one finds the following bounds for $\hat{\Pi}(\hat{k})$

$$\frac{\pi}{\hat{k}} \arctan \frac{\hat{k}}{2\sqrt{1 + \hat{\Pi}_{max}}} \leq \frac{\hat{\Pi}(\hat{k})}{\kappa} \leq \frac{\pi}{\hat{k}} \arctan \frac{\hat{k}}{2} \quad (9)$$

which in turn determine the asymptotic behavior, $\hat{\Pi}(\hat{k}) = \pi^2\kappa/(2\hat{k}) + \mathcal{O}(\hat{k}^{-2})$.

Secondly, we note that due to the z -integration, odd powers of z vanish in the integrand, which in particular means that $\hat{\Pi}'(0) = 0$.

Using this property and the asymptotic behavior, we are already able to approximate $\hat{\Pi}(\hat{k})$ by a suitably chosen function. Due to the non-local character of the integral equation, however, we have to know the function $\hat{\Pi}(\hat{k})$ for all momenta even if we were only interested in the low momentum behavior of the self-energy. Consequently, each kind of series ansatz is not a clever choice since it can never exhibit both, the correct small momentum behavior and the large \hat{k} limit. We thus employ a Padé (rational function) approximation which will turn out to be a remarkably good global approximation for Π . The simplest ansatz consistent with $\hat{\Pi}'(0) = 0$ and the asymptotic behavior reads

$$\hat{\Pi}(\hat{k}) = \frac{A^2 + \frac{\pi^2}{2}\kappa\hat{k}}{a^2 + \frac{a^2}{A^2}\frac{\pi^2}{2}\kappa\hat{k} + \hat{k}^2}, \quad A = A(\kappa), \quad a = a(\kappa) \quad (10)$$

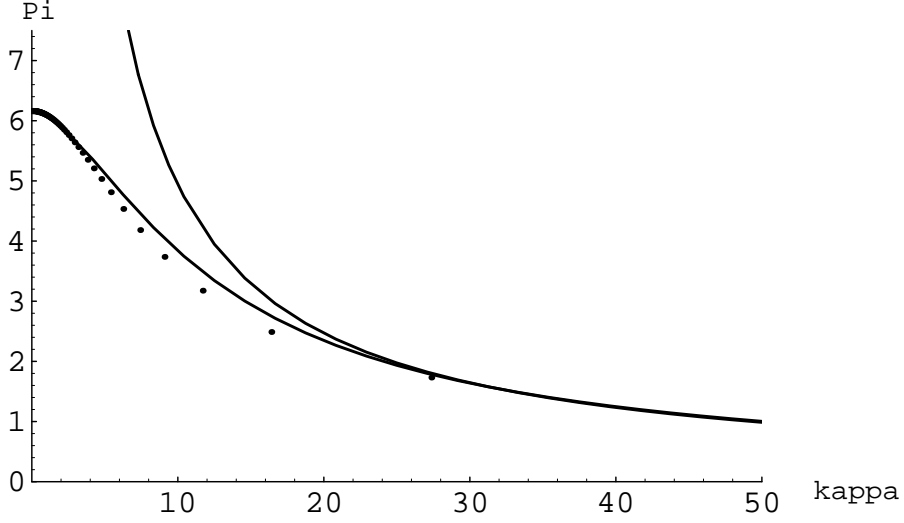


FIG. 2. Solution of the integral equation for $\hat{\Pi}$ at $\kappa = 10$. The dots represent values found by a 64-point Gaussian quadrature of the exact integral equation. The lower rigid line corresponds to the analytic approximation of the self-energy employed throughout this paper. We also indicate the exact asymptotic behavior of $\hat{\Pi}$ (upper line).

and a, A are functions of κ which we have to determine such that the approximation fits as well as possible the true solution.

From the form of the denominators in the integrand (8) it is evident, that $\hat{\Pi}(\hat{p})$ contributes most to the integral for small values of \hat{p} since for large values of \hat{p} , $\hat{\Pi}$ being a decreasing function is neglectable compared with \hat{p}^2 . For this reason we choose the ansatz such that (8) is fulfilled exactly for $\hat{\Pi}(0)$ and $\hat{\Pi}''(0)$. Plugging (10) into the integral equation, one could — at least in principle — carry out the integration which results in two complicated transcendental equations for a, A which have to be solved numerically anyway. We, however, will not do so, but merely discuss the limits $\kappa \rightarrow 0$ and $\kappa \rightarrow \infty$. It turns out (see appendix) that for large values of κ , the coefficients exhibit a power law behavior, namely $A(\kappa) \sim A_0 \kappa^{2/3}$, $a(\kappa) \sim a_0 \kappa^{1/3}$ with the constants $A_0 = 4,8611\dots$ and $a_0 = 4,1140\dots$. For small κ we find $A(\kappa) \sim \sqrt{6\pi}\kappa^{1/2}$ and $a(\kappa) \sim \sqrt{12} + \dots$. Values between the small κ and large κ region can be found by numerical integration, see also the table given in the appendix. In Fig. 2 we compare the approximation (10) with the exact one found by numerical integration of Eq. (8).

The solution for $\hat{\Pi}$ may further serve to eliminate the parameter M as defined in (2), which amounts to evaluate the trace of the propagator,

$$T \sum_{\omega} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{p^2 + \omega^2 + \mu^2 + \Pi(p, \omega)}, \quad (11)$$

where as before, we have to extract the UV-divergent part prior to be able to make the static approximation $\omega = 0$ in the sum. Again subtracting and adding the UV-dominant contribution

$$T \sum_{\omega} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{p^2 + \omega^2 + \mu^2} = \frac{T^2}{12} - \frac{\mu T}{4\pi} + \frac{\Lambda^2}{8\pi^2} + \mu^2 \mathcal{D}, \quad (12)$$

(11) - (12) is a UV-finite quantity, and we are thus allowed to drop all modes except for the zero-mode in the thermal sum. The difference may be expressed by the function

$$\mu F(\kappa) = \frac{2}{\pi} \int_0^{\infty} dp \frac{p^2 \Pi(p)}{(p^2 + \mu^2)(p^2 + \mu^2 + \Pi(p))} \quad (13)$$

which can be calculated analytically using the approximation for $\hat{\Pi}$ together with the known parameters a and A . We mention that, as before, also this function exhibits a power law behavior for large values of the parameter κ , $F(\kappa) \sim F_0 \kappa^{1/3}$, $F_0 = 0.9963 \dots$, and goes like $F(\kappa) \sim \kappa\pi/6$ for small κ . Finally, putting together the definition of M with the condition (7), one can transform the gap-equation (2) into an equation for the mass parameter μ ,

$$\mu^2 + \frac{\lambda^2 \phi^2}{2} \mathcal{D} = -m^2 + \frac{\lambda}{2} \phi^2 + \frac{\lambda}{2} \left(\frac{T^2}{12} - \frac{\mu T}{4\pi} (1 + F(\kappa)) + \frac{\Lambda^2}{8\pi^2} + \mu^2 \mathcal{D} \right), \quad (14)$$

which is nevertheless meaningless until we absorb the divergent contributions Λ^2 , \mathcal{D} in a redefinition of the bare quantities m , λ and ϕ .

RENORMALIZATION

The renormalization of $\lambda\Phi^4$ -theory is delicate task [10]. It can be shown that carrying out the limit in the regularization parameter, the theory in fact becomes trivial in the sense that the renormalized coupling constant vanishes. For our purposes, however, it is more

adequate to keep the cut-off large but finite, since we want to study $\lambda\Phi^4$ -theory only as a model for the Higgs. Consequently, we express the quantities λ, ϕ, m in redefined ones, but keep in mind the eventual limit $\Lambda \rightarrow \infty$.

We regroup the corresponding terms (14) in the following — admittedly suggestive — way

$$\mu^2(1 - \frac{\lambda}{2}\mathcal{D}) = (-m^2 + \lambda\frac{\Lambda^2}{16\pi^2}) + \frac{\lambda}{2}\phi^2(1 - \lambda\mathcal{D}) + \lambda\left(\frac{T^2}{24} - \frac{\mu T}{8\pi}(1 + F(\kappa))\right)$$

and expect that in the renormalized equation, the coupling constant λ gets replaced by the renormalized one λ_R in the rightmost term. Multiplying the whole equation by λ_R/λ suggests the following identifications

$$1 - \frac{\lambda}{2}\mathcal{D} = \frac{\lambda}{\lambda_R}, \quad -m^2 + \lambda\frac{\Lambda^2}{16\pi^2} = \frac{\lambda}{\lambda_R}m_R^2, \quad \phi^2(1 - \lambda\mathcal{D}) = \phi_R^2. \quad (15)$$

Since we have only calculated to order λ we are free to neglect terms $\mathcal{O}(\lambda^2)$ which have to be fixed by higher order loop calculations. Thus the equation for ϕ may be recast in the more convenient form

$$\phi(1 - \frac{\lambda}{2}\mathcal{D}) = \phi_R$$

which reveals an essential fact. From this equation and the redefinition of the coupling constant λ in (15) we find immediately that $\phi\lambda = \phi_R\lambda_R$ and consequently $\kappa = \kappa_R$. The invariance of κ under the redefinitions is important for the solutions of the gap-equation. As opposed to this, if κ had a non-trivial scaling with a divergent quantity, the integral equation would have only the physically uninteresting solutions for $\kappa = 0$ or $\kappa \rightarrow \infty$ in the limit $\Lambda \rightarrow \infty$. Finally we are able to reexpress the gap-equation for the mass parameter in terms of the redefined quantities,

$$\mu^2 = -m_R^2 + \frac{\lambda_R}{2}\phi_R^2 + \frac{\lambda_R}{2}\left(\frac{T^2}{12} - \frac{\mu T}{4\pi}(1 + F(\kappa))\right). \quad (16)$$

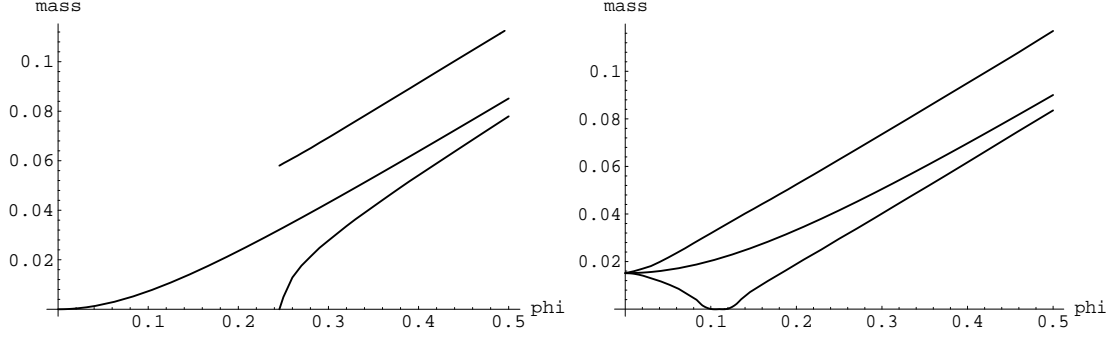


FIG. 3. The mass parameter μ for the temperatures T_c (left) and \hat{T} (right) for $\lambda_R = 0.1$ in units of m_R . The lowest curve corresponds to the solution μ of the mass equation. The middle curve are the values found without the correction term $F(\kappa)$ (superdaisy resummation) and the upper curve corresponds to the 'infrared mass'.

DISCUSSION

The mass equation (16) — together with the known function $F(\kappa)$ — may be used to determine the effective mass parameter μ for a given value ϕ_R . First we note that the contribution $F(\kappa)$ entirely comes from the resummation of the second graph in Fig. 1. For small values of the parameter κ , the additional term can be neglected with respect to 1. On the other hand, for large values of κ which in the non-trivial vacuum $\phi_R \neq 0$ means small μ , due to the large κ behavior of $F(\kappa) \sim \kappa^{1/3}$, the mass parameter cancels out in the next to leading order contribution in equation (16) and we are left with a term $\mu T F(\kappa) \propto T^{4/3}(\lambda_R \phi_R)^{2/3}$. We emphasize that this term is an important contribution near the critical temperature $T_c = m_R \sqrt{24/\lambda_R}$ because it takes on a non-vanishing value for $\mu \rightarrow 0$. In particular, expressed in orders of λ_R , since $\phi_{R,min} = \mathcal{O}(1)$ one finds $\lambda_R \mu T_c F(\kappa) \sim \lambda_R T_c^{4/3} (\lambda_R \phi_R)^{2/3} = \mathcal{O}(\lambda_R)$ which is of the same order as the term $\lambda_R \phi_R^2/2$ and thus crucial at the critical temperature where the leading term $-m_R^2 + \lambda_R T_c^2/24$ in (16) vanish.

Due to this behavior the mass-equation does not necessarily have a real solution for all values ϕ_R . In particular, at T_c no solutions exists for $\phi_R \leq m_R \sqrt{3} F_0^{3/4} / (8\pi^5)^{1/4} = 0.245 \dots$. For temperatures between T_c and $\hat{T} = T_c (1 - \lambda_R F_0^{3/2} / \sqrt{24\pi^5})^{-1}$, the solution consists of two disconnected branches which join for $T \geq \hat{T}$ to give a solution for all values of ϕ_R (Fig. 3).

The mass parameter may be interpreted physically as a kind of 'ultraviolet' mass appearing in the propagator (3) since the self-energy vanishes for large momenta. As opposed to this, in the infrared region $k \rightarrow 0$ the self-energy does have the non-vanishing limit $\mu^2 A^2/a^2$ which contributes to the mass in the propagator. The corresponding effective masses are shown in Fig. 3.

CONCLUSION

We have solved the gap-equation consistently to lowest order for a spontaneously broken theory. It turns out, that the Hartree-Fock approximation corresponding to superdaisy resummation fails to be consistent. Instead, for small effective mass, we encountered an additional contribution in the gap-equation which behaves like $\phi^{2/3}$ which cannot be found in a perturbative calculation. This contribution of course alters the effective propagator and consequently the corresponding effective potential. Since it is exactly the region of small mass which is of crucial importance for the behavior of a theory near the phase transition, we expect significant changes in the transition of field configurations from the false to the trivial vacuum and vice versa. We plan to clarify this point in a forthcoming investigation.

APPENDIX:

The conditions imposed to determine a, A are that equation (8) is fulfilled exactly for $\hat{\Pi}(0)$ and $\hat{\Pi}''(0)$. From the first condition, we find, putting the ansatz (10) into (8)

$$\frac{A^2}{a^2} = \kappa \int_{-1}^1 dz \int_0^\infty d\hat{p} \frac{\hat{p}^2}{N^2}, \quad N = \hat{p}^2 + 1 + \frac{A^2 + \frac{\pi^2}{2} \kappa \hat{p}}{a^2 + \frac{a^2}{A^2} \frac{\pi^2}{2} \kappa \hat{p} + \hat{p}^2}. \quad (\text{A1})$$

The functions a, A can be approximated consistently for large values of κ by a simple power law, $A(\kappa) \sim A_0 \kappa^{2/3}$, $a(\kappa) \sim a_0 \kappa^{1/3}$. To see this explicitly, we plug the assumed behavior into equation (A1) and rescale the momentum by $\hat{p} = \kappa^{1/3} x$ which transforms the denominator N into

$$N = 1 + \kappa^{2/3} L, \quad L = \left(x^2 + \frac{A_0^2 + x \frac{\pi^2}{2}}{a_0^2 + x \frac{\pi^2 a_0^2}{2 A_0^2} + x^2} \right).$$

For large values of κ we neglect 1 with respect to the term containing L . Thus the integral becomes proportional to $\kappa^{-1/3}$ which is consistent with the power-behavior of A^2/a^2 and (A1) takes the form

$$\frac{A_0^2}{a_0^2} = 2 \int_0^\infty dx \frac{x^2}{L^2}. \quad (\text{A2})$$

Similarly, from the second condition for $\hat{\Pi}''(0)$, one finds

$$\frac{A_0^2}{a_0^4} = \int_0^\infty dx \left(\frac{x^2(3 + 2L_2 + L_3)}{3L^3} - \frac{2x^4(2 + L_2)^2}{3L^4} \right), \quad (\text{A3})$$

$$L_2 = -\frac{2A_0^4(4A_0^2 + \pi^2 x)}{(2a_0^2 A_0^2 + a_0^2 \pi^2 x + 2A_0^2 x^2)^2}, \quad L_3 = \frac{8A_0^6(-2a_0^2 A_0^2 + 6A_0^2 x^2 + x^3 \pi^2)^2}{(2a_0^2 A_0^2 + a_0^2 \pi^2 x + 2A_0^2 x^2)^3}.$$

The set of equations (A2,A3) can be solved numerically for the constants A_0, a_0 , and we find $a_0 = 4.1140123\dots$, $A_0 = 4.8610750\dots$.

The function $F(\kappa)$ defined in (13) may be calculated analogously by the same reasoning as above to have the asymptotic form $F(\kappa) \sim F_0 \kappa^{1/3}$ where the constant $F_0 = 0.996322285\dots$ is given by

$$F_0 = \frac{2}{\pi} \int_0^\infty dx \frac{L - x^2}{L}$$

For vanishing parameter κ , by the bounds given in (9), we have $\Pi_{max} \sim \kappa$ and consequently

$$\hat{\Pi}(\hat{k}) = \kappa \frac{\pi}{\hat{k}} \arctan \frac{\hat{k}}{2}$$

Reexpressed in the functions a, A this means $A(\kappa) \sim \sqrt{6\pi}\kappa^{1/2}$, $a(\kappa) \sim \sqrt{12}$ and $F(\kappa) \sim \pi\kappa/6$.

Further values may be read off from the following table.

κ	10^{-2}	10^{-1}	1	10	10^2	10^3	10^4	10^5	10^6
a	3.491	3.709	4.995	9.098	19.24	41.58	89.56	192.9	415.6
A	0.4358	1.421	5.210	22.31	104.9	493.0	2291	10636	49370
F	0.005397	0.04984	0.3287	1.293	3.701	9.043	20.59	45.49	99.15

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